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Micropolar elasticity by unified theory

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Unified theory of structures based on micropolar elasticity

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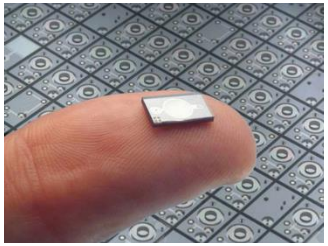
Presentation outline

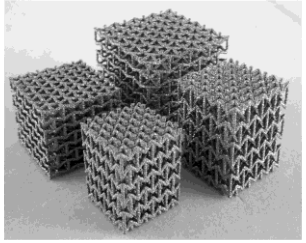
- Introduction and state-of-the-art
- Micropolar Elasticity
- Carrera Unified Formulation (**CUF**)
- Unified Formulation of Micropolar Elasticity
- Numerical results
- *Conclusions*

Structures

Large scale

Smaller scale





- Classical theories
- Higher-order theories

- Continuum mechanics classical elasticity
- Nonlocal elasticity (continuum and discrete)

State-of-the-art

Independent couple stress vector added to the classical couple stress vector («couple-stress elasticity»).

W. Voigt,
"Theoretische Studien über die Elasticitätsverhältnisse der Krystalle",
1887.

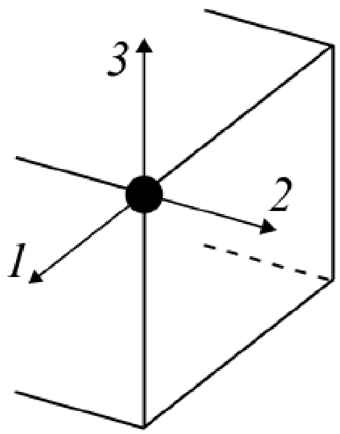
Independent displacement and microrotation field vectors.
Six degrees of freedom for every element.

E. Cosserat, F. Cosserat,
"Théorie des corps déformables",
1909.

Further development of Cosserat theory of elasticity («micropolar elasticity»).

A. C. Eringen,
"Linear theory of micropolar elasticity",
1966.

Micropolar Elasticity



Force stress

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

Strain

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{21} & \varepsilon_{31} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{32} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

Couple stress

$$\mu = \begin{bmatrix} \mu_{11} & \mu_{21} & \mu_{31} \\ \mu_{12} & \mu_{22} & \mu_{32} \\ \mu_{13} & \mu_{23} & \mu_{33} \end{bmatrix}$$

Twist

$$\chi = \begin{bmatrix} \chi_{11} & \chi_{21} & \chi_{31} \\ \chi_{12} & \chi_{22} & \chi_{32} \\ \chi_{13} & \chi_{23} & \chi_{33} \end{bmatrix}$$

Unknowns

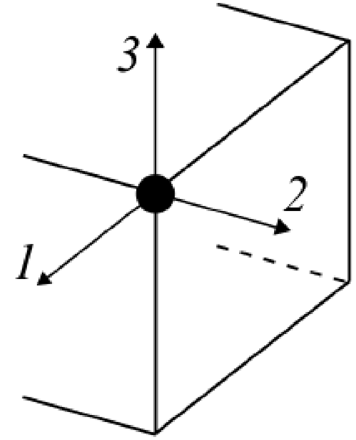
$$\boldsymbol{u}_{(1,2,3)} = \{u_1 \quad u_2 \quad u_3 \quad \omega_1 \quad \omega_2 \quad \omega_3\}^T$$

Displacement

Micro-Rotation

Strain-displacement relations:

- $\varepsilon_{ij} = u_{i,j} + e_{ijk}\omega_k$ $\quad$ $i,j,k = 1,2,3$
- $\chi_{ij} = \omega_{i,j}$



Constitutive relations:

- $\sigma = \lambda(tr\varepsilon)I + (\mu+\alpha)\varepsilon + (\mu-\alpha)\varepsilon^T$ $\quad$ *λ and μ are traditional Lame's constants*
- $\mu = \beta(tr\chi)I + (\gamma+\varepsilon)\chi + (\gamma-\varepsilon)\chi^T$ $\quad$ *α, β, γ and ε are extra micropolar elastic constants*

The Unified Formulation allows for straightforward implementation of any-order beam theories

Main Features

- *Hierarchical formulation*, i.e. the order of the theory can be set as an input of the analysis.
- *Different expansion* types can be adopted (e.g. Taylor, Lagrange).
- *Arbitrary geometries and boundary conditions* can be used.

Unified Formulation of the Displacement Field

$$\mathbf{u}(x, y, z) = F_\tau(x, z)\mathbf{u}_\tau(y), \qquad \tau = 1, 2, ..., M$$

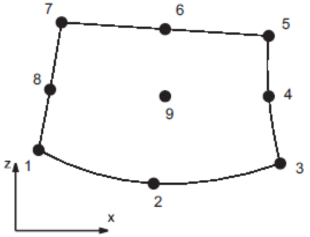
F_τ = $f(x, z)$, where x and z are the cross-section coordinates

In LE CUF models, Lagrange polynomials are used to *develop beam theories*.

$$\boldsymbol{u}(x,y,z) = F_\tau(x,z)\,\boldsymbol{u}_\tau(y)$$

*Three- (L3), four- (L4), and nine-point (L9) Lagrange polynomials can be used which lead to linear, quasi-linear (bilinear), and quadratic displacement field approximations.

E.g., **quadratic** Lagrange polynomial (**L9**).



$$F_\tau = \frac{1}{4}(r^2 + r\,r_\tau)(s^2 + s\,s_\tau), \quad \tau = 1,3,5,7$$

$$F_\tau = \frac{1}{2}s_\tau^2(s^2 - s\,s_\tau)(1-r^2) + \frac{1}{2}r_\tau^2(r^2 - r\,r_\tau)(1-s^2), \quad \tau = 2,4,6,8$$

$$F_\tau = (1-r^2)(1-s^2), \quad \tau = 9$$

E. Carrera and M. Petrolo, ***"Refined Beam Elements with only Displacement Variables and Plate/Shell Capabilities"*, Meccanica, 47(3), 537–556, 2012.**

CUF + FEM

$$\boldsymbol{u}(x,y,z) = F_\tau(x,z)N_i(y)\boldsymbol{q}_{\tau i} \quad \tau = 1,\ldots,M \quad \text{and} \quad i = 1,\ldots,p+1$$

Unified Formulation of Micropolar Elasticity

- $\varepsilon_{ij} = u_{i,j} + e_{ijk}\omega_k$

$$\downarrow$$

- $\varepsilon = (\boldsymbol{b}_{m1})\boldsymbol{u}$

$$
\mathrm{b}_{m1} = \begin{bmatrix}
\partial_x & 0 & 0 & 0 & 0 & 0 \\
0 & \partial_y & 0 & 0 & 0 & 0 \\
0 & 0 & \partial_z & 0 & 0 & 0 \\
\partial_y & 0 & 0 & 0 & 0 & 1 \\
0 & \partial_x & 0 & 0 & 0 & -1 \\
\partial_z & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & \partial_x & 0 & 1 & 0 \\
0 & \partial_z & 0 & 1 & 0 & 0 \\
0 & 0 & \partial_y & -1 & 0 & 0
\end{bmatrix}
$$

CUF
FEM

$$
\mathbf{B}_{m1}^{sj} = \begin{bmatrix}
F_{s,x}N_j & 0 & 0 & 0 & 0 & 0 \\
0 & F_sN_{j,y} & 0 & 0 & 0 & 0 \\
0 & 0 & F_{s,z}N_j & 0 & 0 & 0 \\
F_sN_{j,y} & 0 & 0 & 0 & 0 & F_sN_j \\
0 & F_{s,x}N_j & 0 & 0 & 0 & -F_sN_j \\
F_{s,z}N_j & 0 & 0 & 0 & -F_sN_j & 0 \\
0 & 0 & F_{s,x}N_j & 0 & F_sN_j & 0 \\
0 & F_{s,z}N_j & 0 & F_sN_j & 0 & 0 \\
0 & 0 & F_sN_{j,y} & -F_sN_j & 0 & 0
\end{bmatrix}
$$

Unified Formulation of Micropolar Elasticity

- $\chi_{ij} = \omega_{i,j}$

$$\downarrow$$

- $\chi = (\boldsymbol{b}_{m2})\boldsymbol{u}$

$$\mathbf{b}_{m2} = \begin{bmatrix} 0 & 0 & 0 & \partial_x & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_y & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_z \\ 0 & 0 & 0 & \partial_y & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_x & 0 \\ 0 & 0 & 0 & \partial_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_x \\ 0 & 0 & 0 & 0 & \partial_z & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_y \end{bmatrix}$$

CUF
FEM

$$\mathbf{B}^{sj}_{m2} = \begin{bmatrix} 0 & 0 & 0 & F_{s,x}N_j & 0 & 0 \\ 0 & 0 & 0 & 0 & F_s N_{j,y} & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{s,z}N_j \\ 0 & 0 & 0 & F_s N_{j,y} & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{s,x}N_j & 0 \\ 0 & 0 & 0 & F_{s,z}N_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{s,x}N_j \\ 0 & 0 & 0 & 0 & F_{s,z}N_j & 0 \\ 0 & 0 & 0 & 0 & 0 & F_s N_{j,y} \end{bmatrix}$$

Unified Formulation of Micropolar Elasticity

- $\sigma = \lambda(tr\varepsilon)I + (\mu+\alpha)\varepsilon + (\mu-\alpha)\varepsilon^T \longrightarrow \sigma = \mathbf{C}\,\varepsilon$
- $\mu = \beta(tr\chi)I + (\gamma+\varepsilon)\chi + (\gamma-\varepsilon)\chi^T \longrightarrow \mu = \mathbf{A}\,\chi$

$$\mathbf{C} = \begin{bmatrix} C_1 & C_2 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2 & C_1 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2 & C_2 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_3 & C_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_4 & C_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_3 & C_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_4 & C_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 & C_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_4 & C_3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} A_1 & A_2 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_2 & A_1 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_2 & A_2 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_3 & A_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & A_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_3 & A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_4 & A_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_3 & A_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & A_3 \end{bmatrix}$$

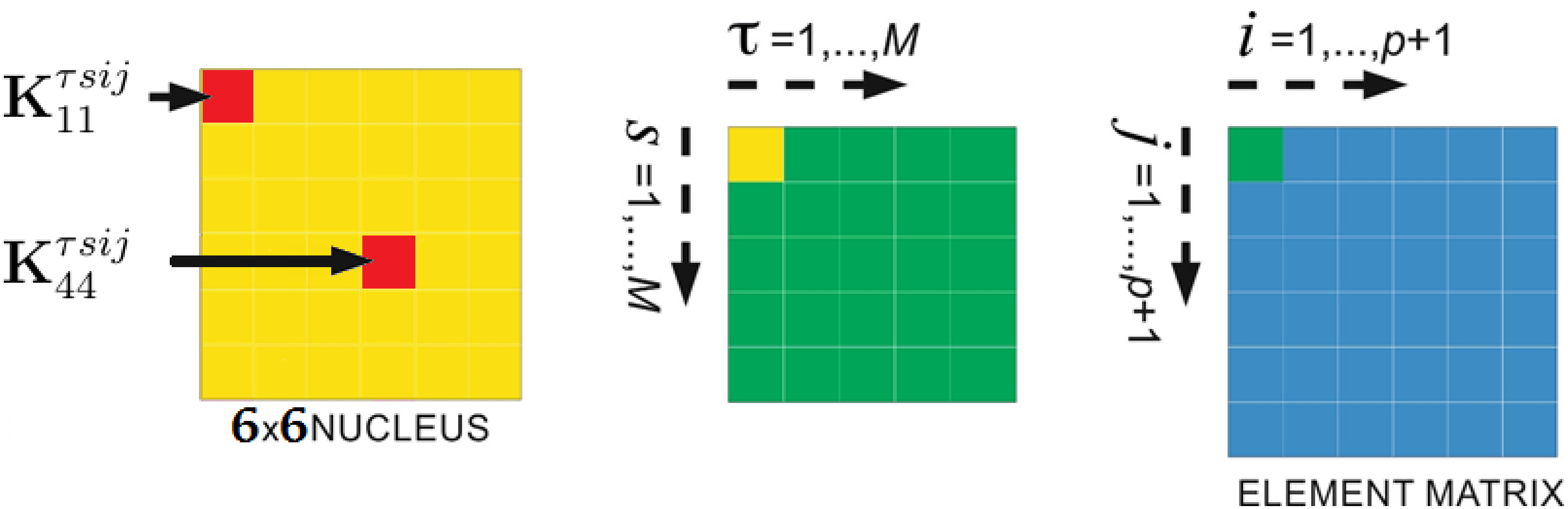
$$C_1 = \lambda + 2\mu, \quad C_2 = \lambda, \quad C_3 = \mu + \alpha, \quad C_4 = \mu - \alpha \qquad A_1 = \beta + 2\gamma, \quad A_2 = \beta \quad A_3 = \gamma + \epsilon, \quad A_4 = \gamma - \epsilon$$

$$\delta L_{\text{int}} = \int (\delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma}) dV = \delta \mathbf{q}_{\tau i}^T \int ((\mathbf{B}_{m1}^{\tau i})^T \mathbf{C}\, (\mathbf{B}_{m1}^{sj}) + (\mathbf{B}_{m2}^{\tau i})^T \mathbf{A}\, (\mathbf{B}_{m2}^{sj})) dV\, \mathbf{q}_{sj} = \delta \mathbf{q}_{\tau i}^T \mathbf{K}^{ij\tau s} \mathbf{q}_{sj}$$

$$\boxed{\mathbf{K}_{11}^{\tau sij}} = C_1 \int_A F_{\tau,x} F_{s,x} dxdz \int_l N_i N_j dy + C_3 \left(\int_A F_{\tau,z} F_{s,z} dxdz \int_l N_i N_j dy + \int_A F_\tau F_s dxdz \int_l N_{i,y} N_{j,y} dy \right)$$

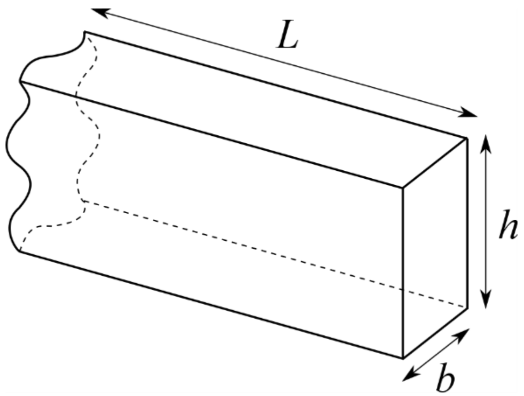
$$\boxed{\mathbf{K}_{44}^{\tau sij}} = 2C_3 \int_A F_s F_t dxdz \int_l N_i N_j dy - 2C_4 \int_l F_s F_t dxdz \int_l N_i N_j dy +$$

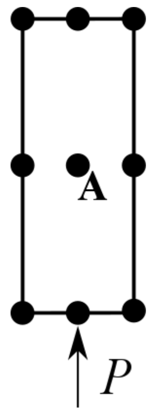
$$A_1 \int_A F_{\tau,x} F_{s,x} dxdz \int_l N_i N_j dy + A_3 \left(\int_A F_{\tau,z} F_{s,z} dxdz \int_l N_i N_j dy + \int_A F_\tau F_s dxdz \int_l N_{i,y} N_{j,y} dy \right)$$



Numerical Results

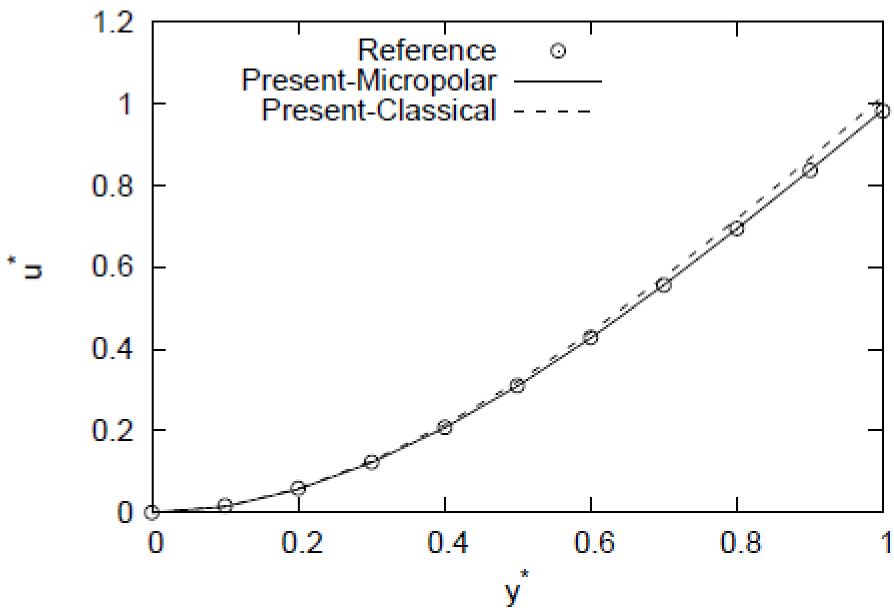
- Cantilever beam subjected to flexure – Ramezani (2009)

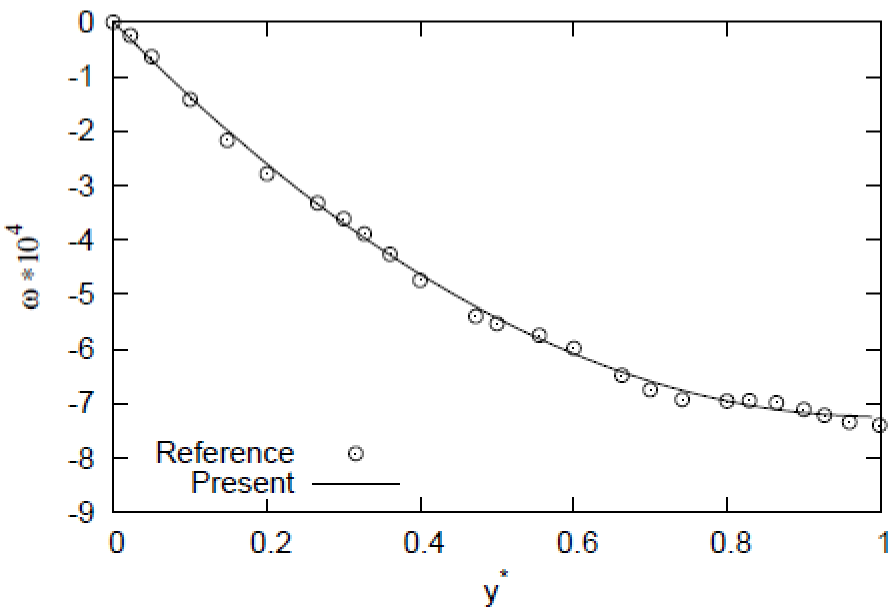




$$\frac{L}{h} = 5 \quad \frac{h}{b} = 4$$

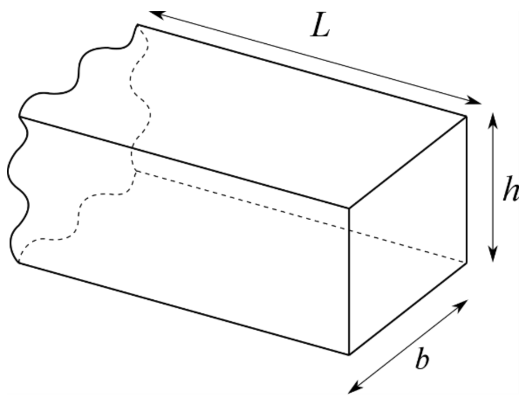
$$E = 20\ GPa, \nu = 0.3, \alpha = \frac{G}{40}, \gamma = \varepsilon = \frac{G}{10000}$$

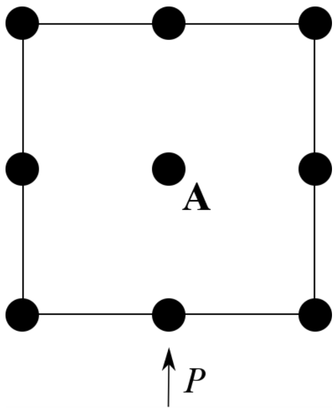




Numerical Results

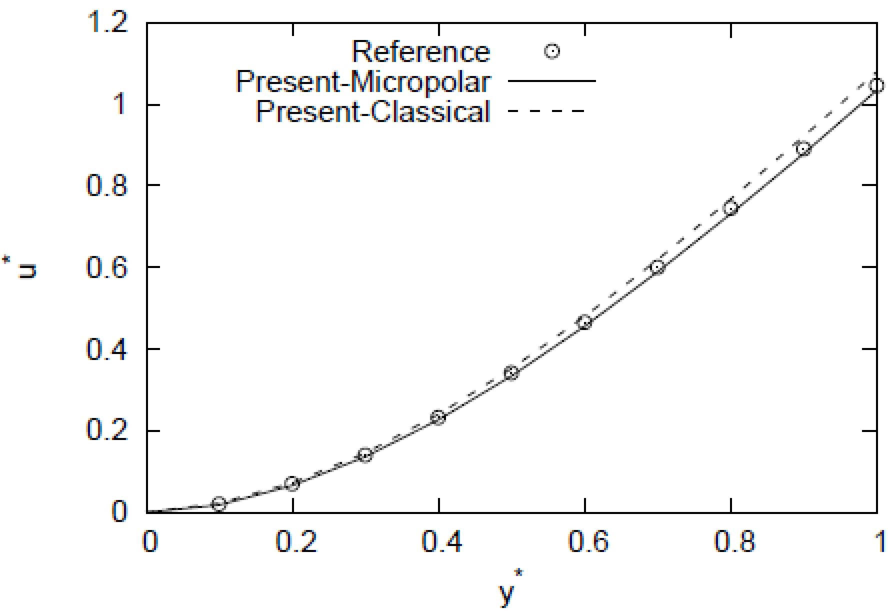
- Cantilever beam subjected to flexure – Hassanpour (2016)

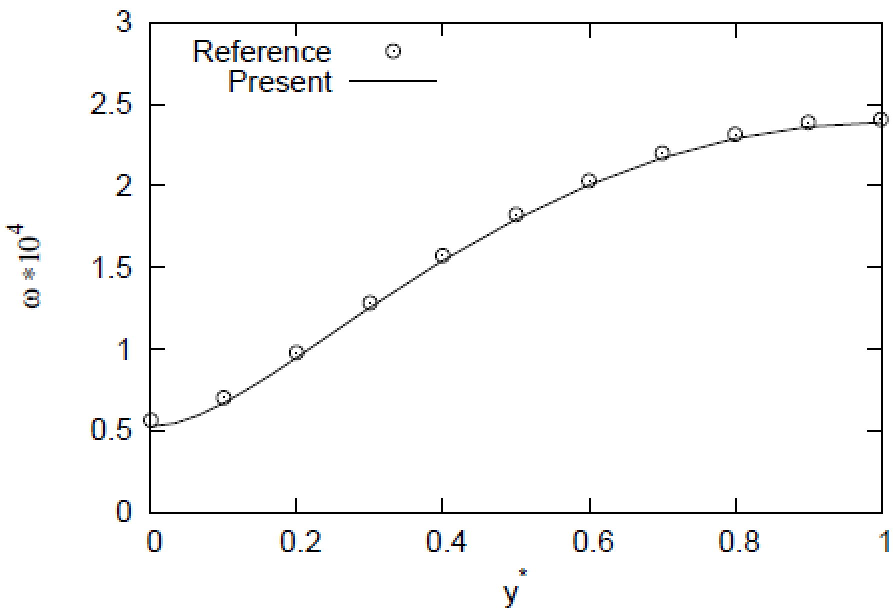




$$\frac{\alpha}{E} = 10^{-2} \qquad \frac{\gamma}{\mathrm{E}} = \frac{\varepsilon}{\mathrm{E}} = 2.5 * 10^{-4}$$

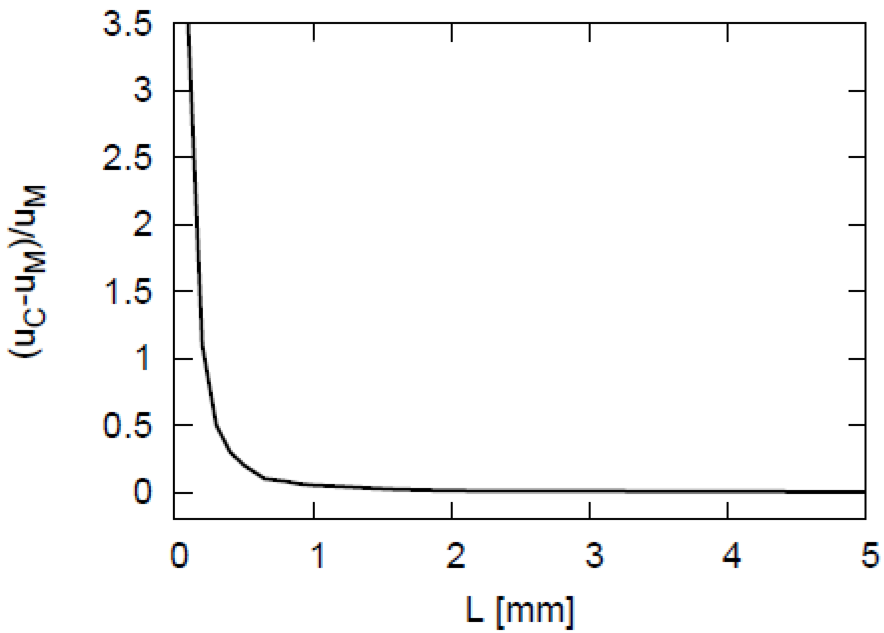
$$\frac{P}{EA} = 5 * 10^{-6} \qquad \frac{L}{\sqrt{\frac{I}{A}}} = 10$$





Numerical Results

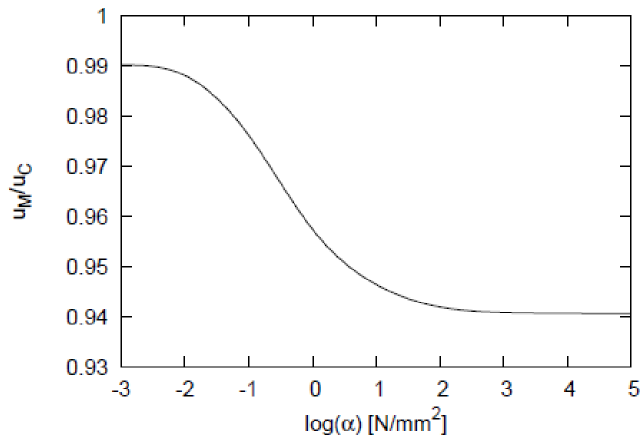
- Size effects

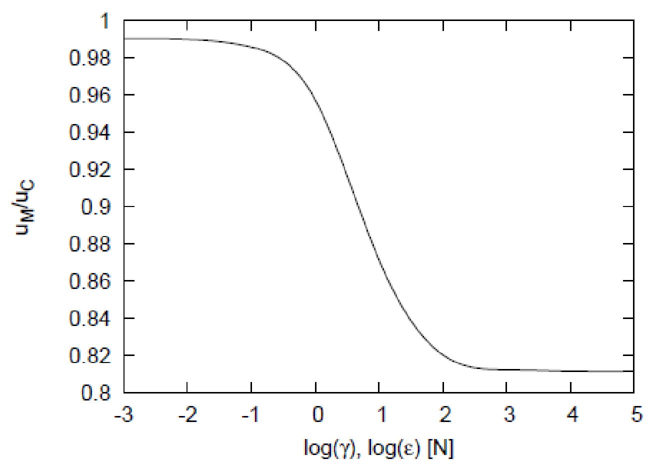


Length	*% diff*
0.1	3.44
0.2	1.08
0.5	0.20
0.8	0.08
1	0.05
1.2	0.04
1.4	0.03
1.6	0.02
2	0.01
5	0.00
10	0.00

- Percentage difference between Micropolar elasticity and Classic elasticity increases when the length of the beam decreases;

- Microrotations have more influence when the length of the beam is small.





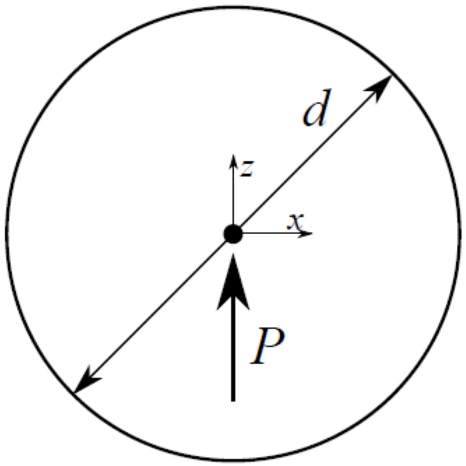
- Micropolar parameter influence on displacement is shown in the figures;

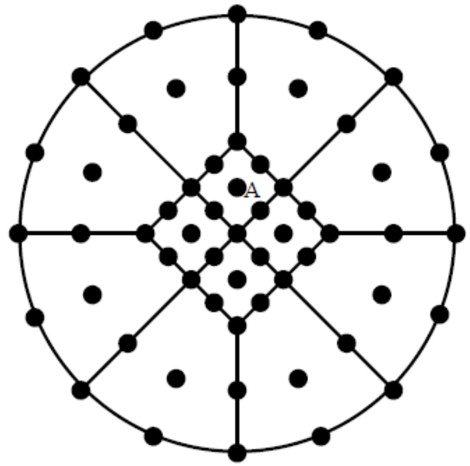
- When the values of the parameters are very low, the micropolar solution is similar to the static one;

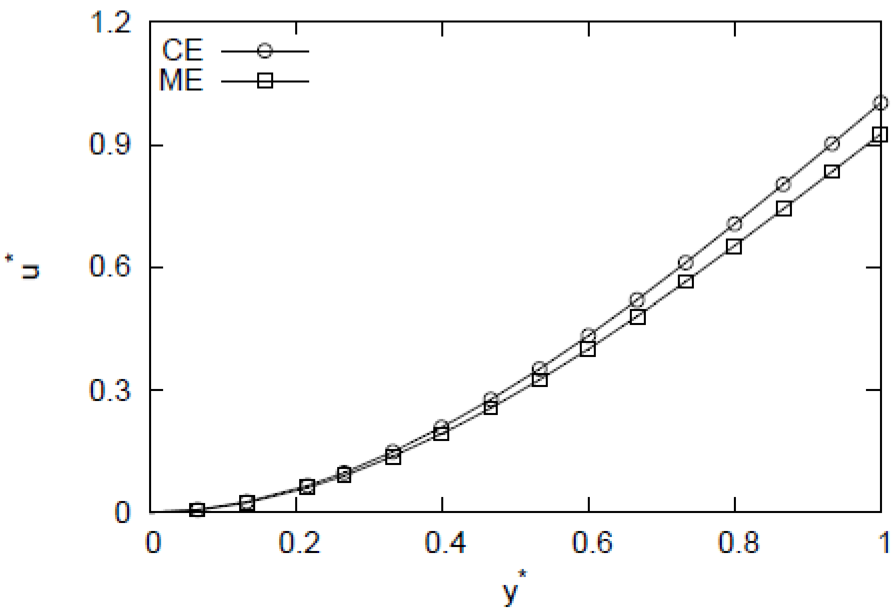
- In each figure, the other parameters (classic and micropolar) are fixed: in this configuration, **a** has less influence than **g** and **e**

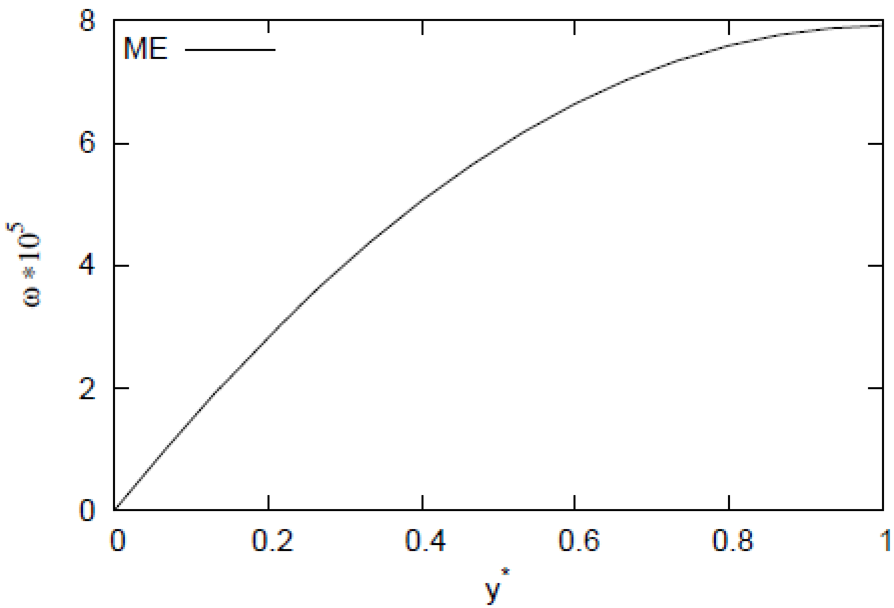
Numerical Results

Human Bone Specimen – Lakes (1991)









Conclusions

- Micropolar elasticity is a theory used when the dimensions of the structure analysed are small. It is based on continuum mechanics, and add a couple stress tensor to the classical force stress tensor, and this affects both kinematic and constitutive relations;

- **CUF** has been demonstrated to provide an efficient formulation for the development of a unified formulation of Micropolar Elasticity, by comparing results with those from literature.

- The results show the importance of using a theory which takes into account the possibility for the nodes of the structure to rotate, like Micropolar elasticity does, for the case of small structures. The influence of Micropolar parameters is shown.

Unified theory of structures based on micropolar elasticity

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